The hard yet tantalizing homology of Out(F_n)

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Outline:

- What's $H_*(Out(F_n))$?
- Group cohomology
- How is $H_*(Out(F_n))$ computed?
- Known non-trivial classes
- Assembly Maps

F_n: free group on n generators

$Aut(G)$: group of isomorphisms from G to itself.

$Inn(G)$: elements of $Aut(G)$ given by $\varphi(g) = hgh^{-1}$ for fixed h.

E.g. $F_2 = \langle a, b \rangle$. define

$\varphi: F_2 \to F_2 \quad w \mapsto awa^{-1}$

$\varphi \in Inn(F_2)$

$\varphi(w_1)\varphi(w_2) = aw_1a^{-1}aw_2a^{-1}$

$= aw_1w_2a^{-1} = \varphi(w_1w_2)$.

Ex: Show $Inn(G)$ is a normal subgroup of $Aut(G)$.

$Out(G)$: $Aut(G)/Inn(G)$.

Group homology

Given a group G, let BG be the classifying space of G. (When G is discrete, this is the path-connected space $\pi_1(BG) \cong G$, $\pi_n(BG) = 0$ $(n \geq 2)$)

$$\underset{\text{grp co.}}{H^k(G;\mathbb{Z})} \cong \underset{\text{singular co.}}{H^k(BG)}$$

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$$= aw_1w_2a^{-1} = \varphi(w_1w_2).$$

<u>Ex</u>: Show $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

$\mathrm{Out}(G)$: $\mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Group homology

Given a group G, let BG be the classifying space of G. (When G is discrete, this is the path-connected space

$$\pi_1(BG) \cong G, \quad \pi_n(BG) = 0 \ (n \geq 2)$$

$$\underset{\text{grp co.}}{H^k(G;\mathbb{Z})} \cong \underset{\text{singular co.}}{H^k(BG)}$$

Ex $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$

$$\pi_n(S^1 \times S^1) = 0 \quad (n \geq 2)$$

$S^1 \times S^1$ is a BG of $\mathbb{Z}^2$.

$$H^k(\mathbb{Z}^2;\mathbb{Z}) \cong H^k(S^1 \times S^1;\mathbb{Z})$$
$$\cong H^k(\mathbb{R}^2/\mathbb{Z}^2;\mathbb{Z})$$

Group cohomological can also be defined algebraically in terms of projective resolutions:

$$H^k(G;M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z};M)$$

$$H_n(G;M) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z};M)$$

How is $H_*(\mathrm{Out}(F_n))$ computed?

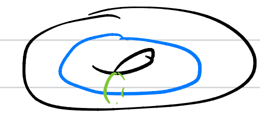
Culler & Vogtmann 86': Defined Outer space X_n, and a subspace K_n, $\mathrm{Out}(F_n) \curvearrowright X_n$, & the action is properly discontinuous and cocompact.

$$H_*(\mathrm{Out}(F_n);\mathbb{Q}) \cong H_*\left(K_n/\mathrm{Out}(F_n);\mathbb{Q}\right)$$

Konstevich: Used spectral sequences to show

$$H_*(\mathrm{Out}(F_n)) \cong \frac{F_p C_p \cap \ker \partial_c \cap \ker \partial_\pi}{\partial_\pi(\ker_c \cap F_{p+1} C_{p+1})}$$

Eg $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$



$\pi_n(S^1 \times S^1) = 0 \quad (n \geq 2)$

$S^1 \times S^1$ is a BG of $\mathbb{Z}^2$.

$$H^k(\mathbb{Z}^2;\mathbb{Z}) \cong H^k(S^1 \times S^1;\mathbb{Z}) \cong H^k(\mathbb{R}^2/\mathbb{Z}^2;\mathbb{Z})$$

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How is $H_*(Out(F_n))$ computed?

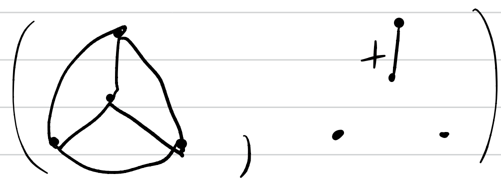
Culler & Vogtmann 86: Defined Outer space X_n, and a subspace K_n, $Out(F_n) \curvearrowright X_n$, & the action is properly discontinuous and cocompact.

$H_*(Out(F_n);\mathbb{Q}) \cong H_*\left(K_n / Out(F_n);\mathbb{Q}\right)$.

Kontsevich: Used spectral sequences to show

$$H_*(Out(F_n)) \cong \frac{F_pC_p \cap \ker\partial_c \cap \ker\partial_R}{\partial_R(\ker_c \cap F_{p+1}C_{p+1})}$$

F_pC_p: pairs (G, Φ) where G is a trivalent graph (vertices have deg 3), $G \simeq \bigvee_n S^1$, and Φ is a collection of p oriented edges in G without cycles.

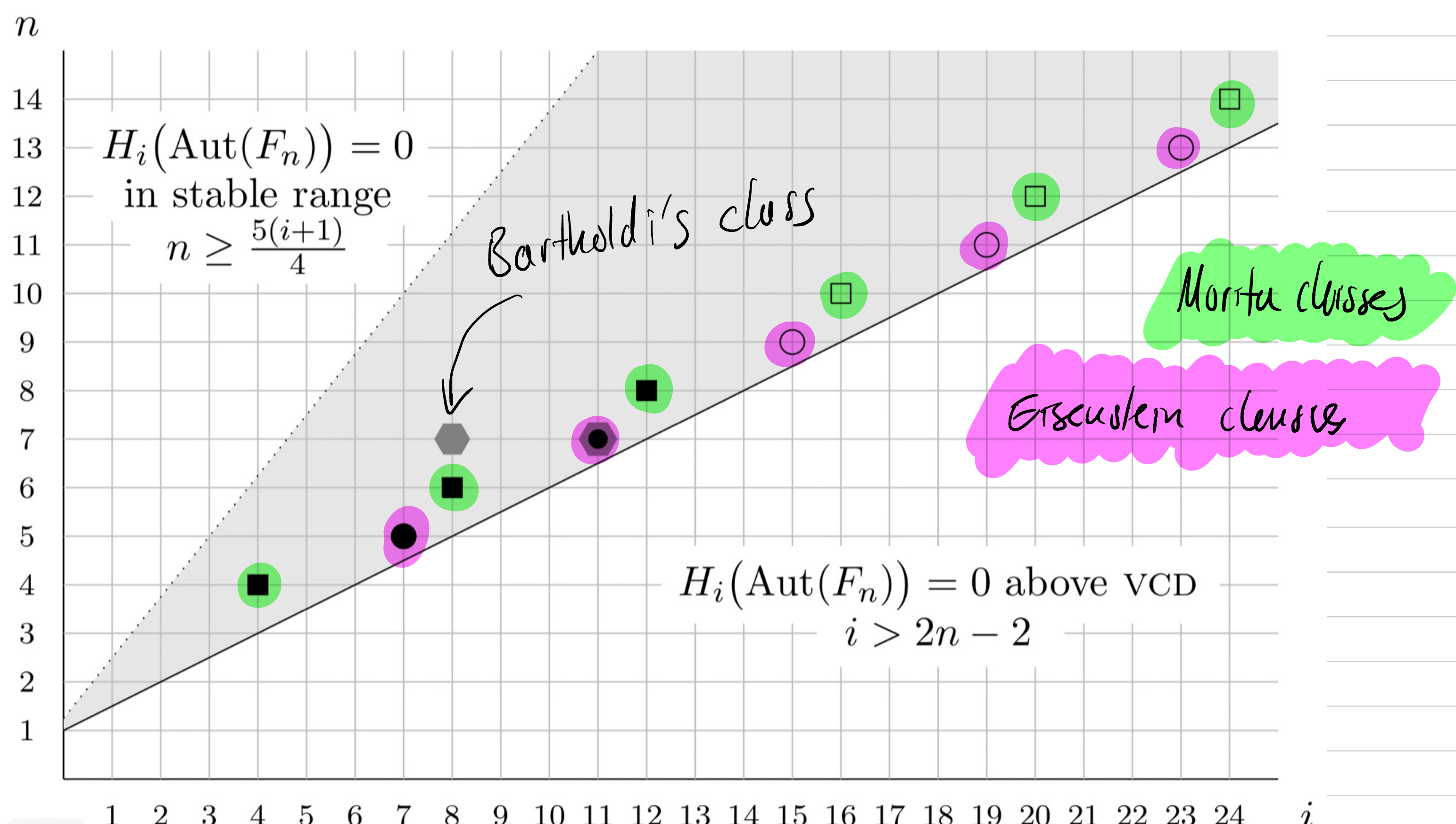
E.g.  $\in C_1 F_1$

$$\partial_c(G,\Phi) = \sum_{e_i \in \Phi} (-1)^i \left(G/e_i, \Phi - e_i\right)$$

$$\partial_R(G,\Phi) = \sum_{e_i \in \Phi} (-1)^i \left(G, \Phi - e_i\right)$$

Known non-trivial classes

(Note: $H_*(\mathrm{Aut}(F_n)) \to H_*(\mathrm{Out}(F_n))$ is surjective.)

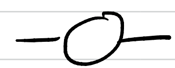


Morita et. al computed:

n	$\chi(Out(F_n))$
3	1
4	2
5	1
6	2
7	1
8	1
9	-21
10	-124
11	-1202

Assembly maps

$X_{n,s}$: rank n graph with s spokes

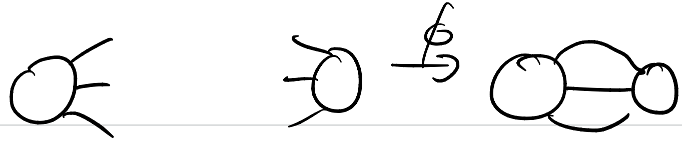
Eg  ,  $= X_{1,2}, X_{2,0}$

$E_{n,s}$: $\{$homotopy equivalences $X_{n,s} \to X_{n,s}$ fix spokes$\}$

$\Gamma_{n,s}$: $\pi_0(\Gamma_{n,s})$

$$Out(F_n) \cong \Gamma_{n,0}$$

$$Aut(F_n) \cong \Gamma_{n,1}$$



$X_{1,3}$ $\qquad$ $X_{1,3}$ $\qquad$ $X_{4,0}$

$\phi : E_{1,3} \times E_{1,3} \to E_{4,0}$

$A\phi : H_*(\Gamma_{1,3}) \otimes H_*(\Gamma_{1,3}) \to H_*(Out(F_4))$

non-trivial

image of the non-trivial gives the first Morita class

→ Eisenstein classes can be obtained like this too.

Assembly maps

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Ex  ,  $= X_{1,2}, X_{2,0}$

$E_{n,s}$: $\{$homotopy equivalences $X_{n,s} \to X_{n,s}$ fix spokes$\}$

$\Gamma_{n,s}$: $\pi_0(\Gamma_{n,s})$

$Out(F_n) \cong \Gamma_{n,0}$

$Aut(F_n) \cong \Gamma_{n,1}$