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Knot polynomials from quantum groups

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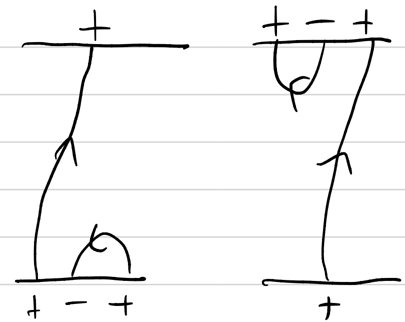
1. Tangle invariants as functors
2. A finite Hopf algebra gives the Jones polynomial.

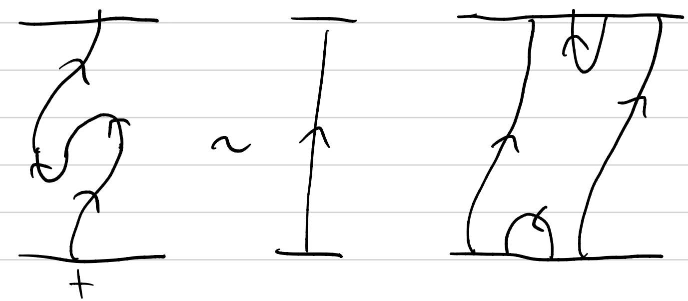
[1] The category Tan has

obj : finite seq. $\pm$ signs

morph: smooth embeddings of oriented 1-manifolds into I^3 with ordered boundary, up to isotopy.



A   B

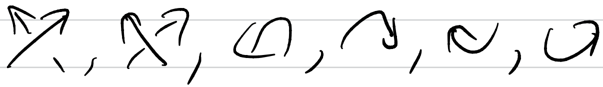


$A \circ B$ $A \otimes B$

<u>Prop:</u> Tan is a tensor-category.

Thm: (Turaev '89) Tan is generated (as a tensor category) by:

obj: $+$, $-$

mor: 

$+$ 8 relations including the three Riedemeister moves.

Def: Let $V \in \mathrm{Vect}$, $c \in \mathrm{Aut}(V \otimes V)$. Then c solves the Yang-Baxter eqn. if

$$(c\otimes 1)(1\otimes c)(c\otimes 1) = (1\otimes c)(c\otimes 1)(1\otimes c)$$

Prop: Solutions to the YB-eqn give* functors $F: \mathrm{Tan} \to \mathrm{Vec}$.

Fact: Solving the YB-eqn is really hard!

$\boxed{2}$

Thm (Drinfeld 87): Braided Hopf algebras give solutions to the YB-eqn.

Thm (Drinfeld): There is a construction which builds braided Hopf algebras from finite Hopf algebras.

Problem: Finite Hopf-algebras which are not commutative, and whose duals are not commutative, give boring tangle invariants... and examples which are neither are hard to come by!

Idea: 'quantum' deformations of universal enveloping algebras to make them non-cocomm.

Fact: $U_q(sl(2))$ is a Hopf algebra, & is neither comm. nor cocomm!

Coro: $U_q(sl(2))$ admits a finite dimensional quotient (and is still a Hopf-algebra).

Thm: $U_q(sl(2))$ is braided.

Prop: $U_q(sl(2))$ gives the solution to the YB-eqn: (dim $V=2$)

$$c = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Exercise: $C - C^{-1} = \left(q - \frac{1}{q}\right) I_4$.

Prop: This gives the skein relation

$$q^2 J\left(\nwarrow\!\!\!\!\!\nearrow\right) - \frac{1}{q^2} J\left(\nwarrow\!\!\!\!\!\nearrow\right) = \left(\frac{1}{q} - q\right) J(\uparrow\uparrow),$$

and hence the Jones polynomial.

Thanks for listening!