Khovanov homology of rational tangles

Benjamin Thompson

Geometry and topology seminar,
Binghamton University,
4/20/23

Khovanov homology of rational tangles

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Cornell University

- Powerful
- Computable

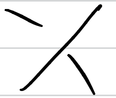
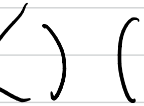
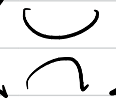
1. Jones Polynomial

Thm: (Jones '87, Kauffman '87)
Let L is an oriented link invariant
with $n^+ := \#$ , $n^- := \#$ .

$\boxed{\bar{q} := q^{-1}}$

Define
$\langle \cdot \rangle : \{\text{link diagrams}\} \to \mathbb{Z}[q, \bar{q}]$

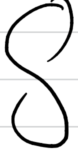
$$\langle \phi \rangle = 1$$
$$\langle L \sqcup O \rangle = (q+\bar{q}) \langle L \rangle$$

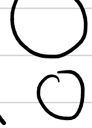
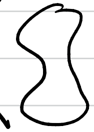
$\langle$  $\rangle = \langle$  $\rangle - q \langle$  $\rangle$.

Then $J(L) := (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

is a link invariant.

i.e. If $L_1 \underset{\text{isot.}}{\simeq} L_2$, $J(L_1) = J(L_2)$.

Eg $J($  $) = q \langle$  $\rangle$

$= q \langle$  $\rangle - q^2 \langle$  $\rangle$

$= q(q+\bar{q})^2 - q^2(q+\bar{q})$

$= \cancel{q}(q+\bar{q})(\cancel{q}+\bar{q}-\cancel{q})$

$= q+\bar{q} = \langle$  $\rangle = J($  $)$.

2 Vanilla Khovanov homology

Notation: Let $V = \bigoplus_{m \in \mathbb{Z}} V_m$

be a graded v.s.

$$q\dim V := \bigoplus_{m \in \mathbb{Z}} q^m \dim V_m$$

$$V\{a\} := \bigoplus_{m \in \mathbb{Z}} V_{m+a}$$

Eg $V = \mathbb{Q}\{X, \mathbb{1}\}$

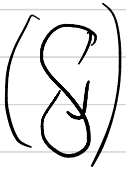
$\deg X = -1$

$\deg \mathbb{1} = 1$

$$q\dim(V^{\otimes b}\{a\}) = q^a(q+\bar{q})^b.$$

$$C = 0 \to V^{\otimes 2}\{1\} \to V\{2\} \to 0$$

$$\chi(C) = \sum_i (-1)^i q\dim C_i$$

$$= q(q+\bar{q})^2 - q^2(q+\bar{q}) = J($$  $$)$$

Q: Can this be done in general?

Thm: (Khovanov, '99)

There is a function
$C : \{\text{link diag.}\} \to Ch(\mathrm{Vect}_k)$

such $L_1 \underset{\text{isot.}}{\simeq} L_2$

$=$ $H_* C(L_1) \simeq H_* C(L_2)$.

$\chi(C(L)) = J(L)$.

Pf: Create giant cube complex, flatten, carefully choose maps between $V^{\otimes b}\{a\}$.

Notation: $Kh(L) := H_* C(L)$.

Rm: $Kh(\cdot)$ is not a complete inv. (Watson '06).

Rm: $Kh(\cdot)$ is strictly stronger than $J(\cdot)$

$$J(5_1) = J(10_{132})$$
$$Kh(5_1) \not\cong Kh(10_{132}).$$

Open problem: Does $J(\cdot)$ detects the unknot?

Thm: (Khronheimer, Mrowka, '10): $Kh(\cdot)$ detects the unknot.

3. Bar-Natan Homology

next page.

Thm (Bar-Natan '05)

There is a category M containing formal sums of tangles, cobordisms with marked points, and a local $[\![\cdot]\!] : \{\text{tangle diagram}\} \to Ch(M)$ such that

$$T_1 \simeq T_2 \Rightarrow [\![T_1]\!] \underset{\substack{\text{chain} \\ \text{htpy}}}{\simeq} [\![T_2]\!].$$

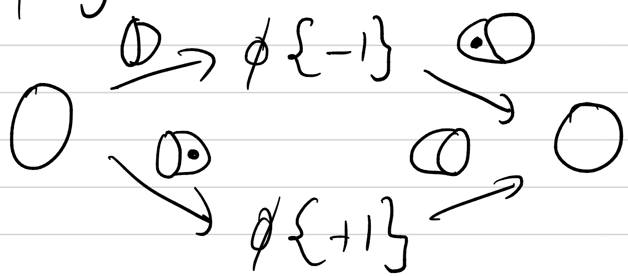
There is a functor $F : M \to \mathrm{Vect}_k$ such that

$$H_* F \circ [\![L]\!] = Kh(L).$$

Rm: Changing F gives different link homologies!

Rm: A variant of F due to Lee gives a combinatorial proof of the Milnor conjecture. (Rasmussen, '04)

Rm M contains isomorphism removing loops from tangles. "delooping"

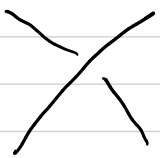
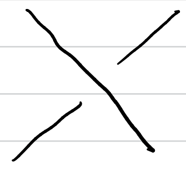


Fact: Delooping & gaussian elimination compute $Kh(L)$ efficiently.

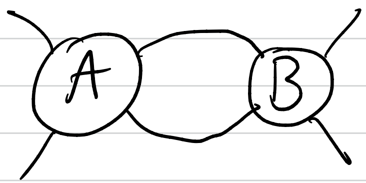
4. Rational tangles

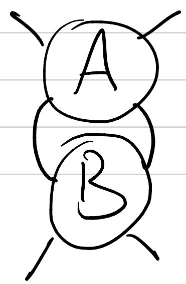
Def A 4-tangle is a smooth embedding of $I \sqcup I \hookrightarrow I^3$ with boundary $\{(0,1/2,0), (1,1/2,0), (0,1/2,1), (1,1/2,1)\}$ considered up to isotopy.

Eg

 $= [1]$  $= [-1]$

Notation: If A, B are 4-tangles, $A+B$, $A\times B$ denote the compositions:

$A+B :=$ 

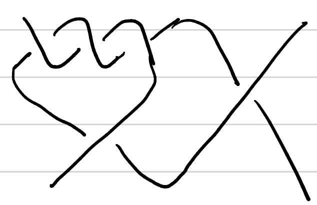
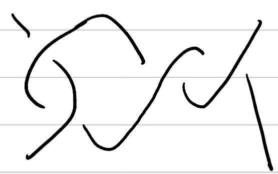
$A\times B :=$ 

$$n^+ = \underbrace{[\pm 1] + \cdots + [\pm 1]}_{n\text{-times}}$$

$$n^\times = \overbrace{[\pm 1] \times \cdots \times [\pm 1]}$$

Def: A 4-tangle is rational if it can be finitely generated from $\{[1], [-1]\}$ under the $+$ and $\times$ operations.

Eg

 $\simeq$ 

$(-3^+ \times 1^\times) + 1^+$ $\qquad$ $(1^+ \times 1^\times) + 2^+$

$$F(T) = 1 + \cfrac{1}{1 + \cfrac{1}{-3}} = 5/2 \qquad F(T) = 2 + \cfrac{1}{1+\cfrac{1}{1}} = 5/2$$

Thm (Conway, '70) For a rational tangle

$$T = [[[(a_1^{+} \times a_2^{\times}) + a_3^{+}] \times a_4^{\times}] + \cdots \times \cdots$$

Define $F(T) = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{+ \cdots + \cfrac{1}{a_1}}}$.

Then $T_1 \simeq T_2$ iff $F(T_1) = F(T_2)$.

5. Bar-Natan homology of rational tangles

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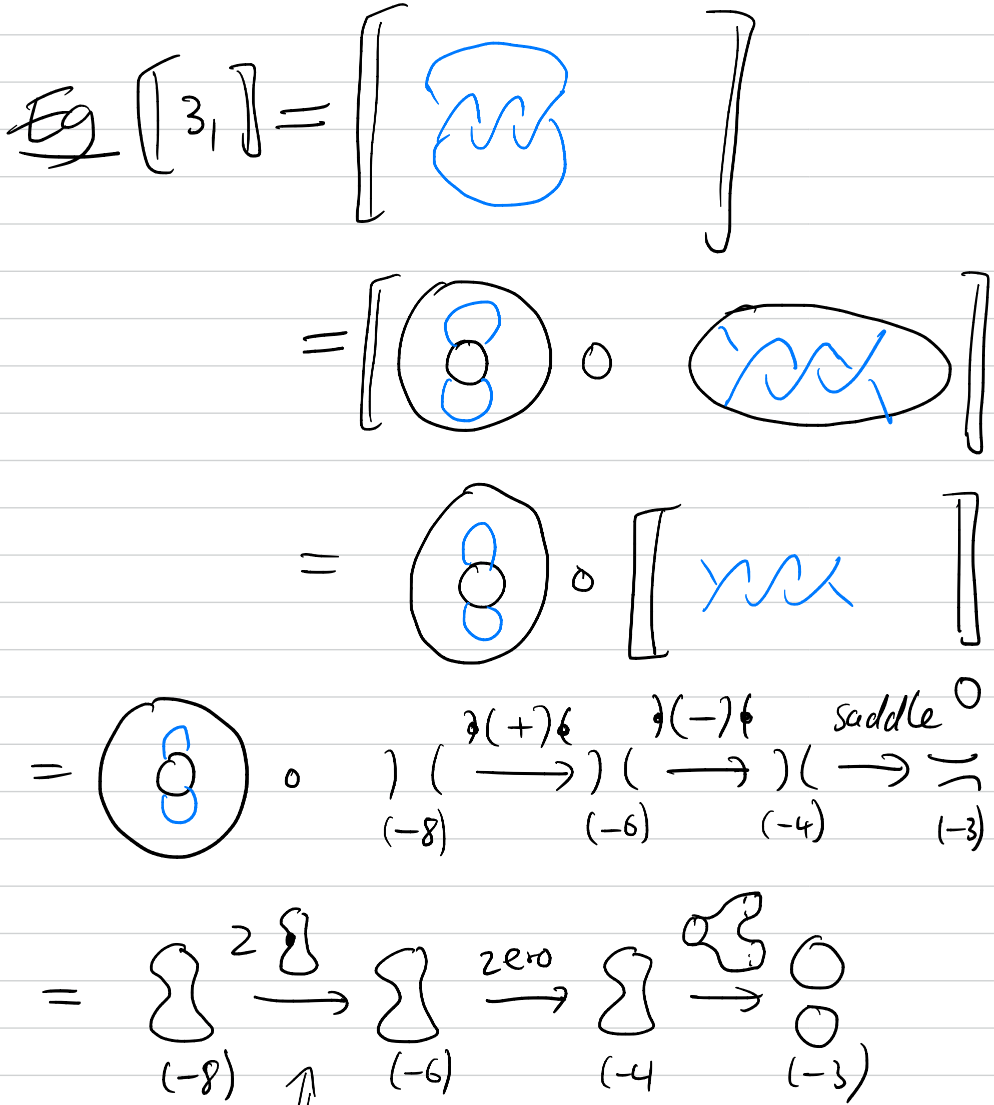

$\Downarrow$ TQFT, H_*

hom. degree

qdim	−3	−2
(−5)		$\mathbb{Z}$
(−7)		
(−9)	$\mathbb{Z}$	$\mathbb{Z}/2$

"knight's move"

	−1	0
−1		$\mathbb{Z}$
−3		$\mathbb{Z}$

"exceptional pair"

Thm (T–, '17) If T is a rational tangle, $[\![T]\!]$ has a rep. with the structure of a zig-zag.

Coro: If L is rational, $Kh(L)$ consists only of knights moves and exactly one exceptional pair.