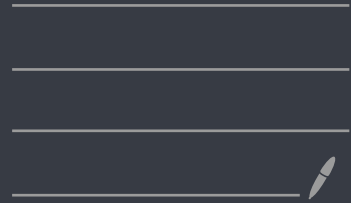


Homotopical Categories I

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- Some constructions

Preserve weak equivalences:

$$\text{Eg } F: \text{Top} \rightarrow \text{Top}$$

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X \times I \\ f & \xrightarrow{\quad} & f \times 1_I \end{array}$$

$$\text{(clearly } f_*: \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$$

$$\text{implies } F(f)_*: \pi_n(X \times I) \xrightarrow{\cong} \pi_n(Y \times I).$$

Eg Let A, B be Abelian categories, & $F: A \rightarrow B$

be additive.

Then F induces a functor

$$F_*: \text{Ch}(A) \rightarrow \text{Ch}(B).$$

If $f \in \text{Mor}(\text{Ch}(A))$ is a chain

homotopy equiv, $F(f) \in \text{Mor}(\text{Ch}(B))$ is too.

Def: (2 of 3 Model Category Axiom)

A collection $W \subseteq \text{Mor}(C)$ satisfies the

two-of-three property if whenever

any two of $\{f, g, gf\}$

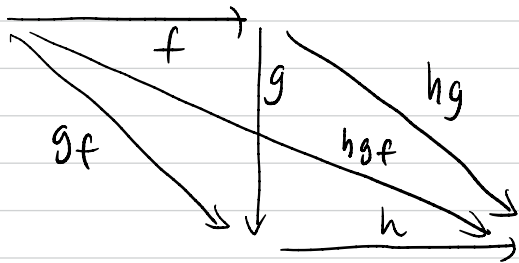
are in W , so is the third.

Eg The isomorphisms of any cat.

Eg Homotopy equivalences in Top.

Def (2 of 6) (Dwyer, Hirschhorn, Kan, Smith '04)

Let $W \subseteq \text{Mor}(\mathcal{C})$. If $\forall f, g, h,$



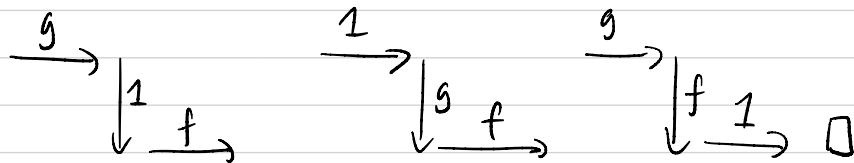
$gf, hg \in W \Rightarrow f, g, h, hgf \in W,$

then W satisfies the two-of-six property -

Rm Weak equivalences in a model cat. satisfy 2 of 6.

Exercise: 2 of 6 \Rightarrow 2 of 3.

Pf: Consider



(DHKS)

Def: Let M be a category, and

$W \subseteq M$ a subcategory. If:

— $\text{obj}(M) = \text{obj}(W)$

— W satisfies 2 of 6,

then (M, W) is called a homotopical category / homotopical.

Eg $(M, \text{Iso}(M))$ is homotopical.

Pf: WTS If hg, gf are isos, so are f, g, h, hgf .

Let $\overline{f_1 \dots f_n}$ denote $(f_1 \dots f_n)^{-1}$, and

$g \in \text{Mor}(x, y)$. Then $g(hgh) = 1_y$,

$$(hgh)_g = (\overline{fg}fg)hgh_g = \overline{fg}f(gghg)g$$

$$= \overline{fg}fg = 1_x,$$

so $g \in \text{Iso}(e)$, and hence

f, h , and gh are too \square

Rm Since $1_x \in W \forall x \in \text{obj}(e)$,

2 of 6 applied to $\begin{array}{c} \xrightarrow{f} \\ \downarrow \overline{f} \\ \xrightarrow{f} \end{array}$ means $\text{Iso}(e) \subseteq W$.

Def: $(M, \text{Iso}(M))$ is called the minimal homotopical category of M .

Eg $(\text{Top}, \{\text{weak equiv}\})$ is homotopical.

Pf: Follows from the functoriality of π_n , and the previous example. \square

Eg $(\text{Top}, \{\text{homotopy equiv}\})$ is homotopical.

Pf: Same proof as with $(M, \text{Iso}(M))$ \square

Def A functor $F: (A, W_A) \rightarrow (B, W_B)$ is homotopical if $F(W_A) \subseteq F(W_B)$.

Eg π_n, H_n .

Non-Example

Let $F: A \rightarrow B$ be additive, A, B abelian, $F_*: \text{Ch}_{\geq 0}(A) \rightarrow \text{Ch}_{\geq 0}(B)$.

Let $W = \{\text{quasi-isomorphism}\}$

$\{f: C \rightarrow D: f_*: H_n(C) \xrightarrow{\cong} H_n(D)\}$

Let $f: C \rightarrow D$ be.

$$\begin{array}{ccccccc}
 \dots & \rightarrow & \mathbb{Z}/2 & \hookrightarrow & \mathbb{Z}/4 & \xrightarrow{0} & \mathbb{Z}/2 & \hookrightarrow & \mathbb{Z}/4 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow & 0 & \rightarrow & \mathbb{Z}/2 & \rightarrow & 0 & \rightarrow & \mathbb{Z}/2 & \rightarrow & 0
 \end{array}$$

Let $F: \text{Ab} \rightarrow \text{Ab}$ be $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$.

Claim: $F(f)$ is not a quasi-iso.

Pf: Board exercise!

(Applying $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ to $\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$ gives $\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$, which after homology does not give an iso.)

Non-Es

Let Top^D be a homotopical category.

Let $\text{colim}: \text{Top}^D \rightarrow \text{Top}$.

Claim: colim is not homotopical.

Pf: Board exercise!

$$\begin{array}{ccccc}
 \text{(Consider} & D^{n+1} & \hookrightarrow & S^n & \hookrightarrow & D^{n+1} \\
 & \downarrow \cong & & \downarrow 1 & & \downarrow \cong \\
 & * & \leftarrow & S^n & \rightarrow & *
 \end{array}$$

$\text{colim}(D^{n+1} \hookrightarrow S^n \hookrightarrow D^{n+1}) = S^{n+1}$, but $\text{colim}(* \leftarrow S^n \rightarrow *) = *$, & $S^n \neq *$.

- We want to find the closest homotopical approximation to an arbitrary $F: (A, W_A) \rightarrow (B, W_B)$.
How do we make one?

- Idea: Kan extensions.

Def Let $F: (A, M_A) \rightarrow (B, M_B)$ be a functor, & $\text{Ho}M, \text{Ho}N$ be the localizations of M, N .

A total left derived functor LF of a functor F is a right Kan extension $\text{Ran}_S F$

$$\begin{array}{ccc}
 M & \xrightarrow{F} & N \\
 \gamma \downarrow & \searrow & \downarrow \delta \\
 \text{Ho}M & \xrightarrow{LF} & \text{Ho}N
 \end{array}$$

Def: A left derived functor of $F: M \rightarrow N$ is a homotopical functor $LF: M \rightarrow N$ and a nat

$\lambda: \mathbb{L}F \rightarrow F$ s.t

$S\lambda: S \cdot \mathbb{L}F \Rightarrow SF$ is a total

left derived functor of F .

(after lifting).

References: Riehl, "Categorical homotopy theory" (2014)