Homotopical Categories I

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• Some constructions preserve weak equivalences:

<u>Eg</u> $F : \mathrm{Top} \to \mathrm{Top}$

$$X \longmapsto X \times I$$
$$f \longmapsto f \times 1_I$$

Clearly $f_* : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ implies $F(f)_* : \pi_n(X \times I) \xrightarrow{\cong} \pi_n(Y \times I)$.

<u>Eg</u> Let A, B be Abelian categories, & $F : A \to B$ be additive.

Then F induces a functor $F_* : Ch(A) \to Ch(B)$.

If $f \in \mathrm{Mor}(Ch(A))$ is a chain homotopy equiv, $F(f) \in \mathrm{Mor}(Ch(B))$ is too.

<u>Def:</u> (2 of 3 Model Category Axiom)
A collection $W \subseteq \mathrm{Mor}(\mathcal{C})$ satisfies the two-of-three property if whenever any two of $\{f, g, gf\}$

are in W, so is the third.

Eg The isomorphisms of any cat.

Eg Homotopy equivalences in **Top**.

Def (2 of 6) (Dwyer, Hirschhorn, Kan, Smith '04)

Let $W \subseteq \mathrm{Mor}(\mathcal{C})$. If $\forall f, g, h,$

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, "hgf"] & \bullet \arrow[d, "g"] \arrow[dr, "hg"] & \\
& \bullet \arrow[r, "h"'] & \bullet
\end{tikzcd}$$

$$gf, hg \in W \Rightarrow f, g, h, hgf \in W,$$

then W satisfies the two-of-six property.

Rm Weak equivalences in a model cat. satisfy 2 of 6.

Exercise: 2 of 6 $\Rightarrow$ 2 of 3.

Pf: Consider

$$\begin{tikzcd}
\bullet \arrow[r, "g"] & \bullet \arrow[d, "1"] & \\
& \bullet \arrow[r, "f"] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "1"] & \bullet \arrow[d, "g"] & \\
& \bullet \arrow[r, "f"] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "g"] & \bullet \arrow[d, "f"] & \\
& \bullet \arrow[r, "1"] & \bullet
\end{tikzcd}
\qquad \square$$

(DHKS)

Def: Let M be a category, and $W \subseteq M$ a subcategory. If:

- $obj(M) = obj(W)$
- W satisfies 2of6,

then (M, W) is called a homotopical category / homotopical.

Eg $(M, Iso(M))$ is homotopical.

Pf: WTS if hg, gf are isos, so are f, g, h, hgf.

Let $\overline{f_1 \cdots f_n}$ denote $(f_1 \cdots f_n)^{-1}$, and $g \in Mor(x, y)$. Then $g(h\overline{gh}) = 1_y$,

$$(h\overline{gh})g = (\overline{fg}fg)h\overline{gh}g = \overline{fg}f(gh\overline{gh})g$$
$$= \overline{fg}fg = 1_x,$$

so $g \in Iso(\mathcal{C})$, and hence f, h, and gfh are too $\square$

Rm Since $1_x \in W$ $\forall x \in obj(\mathcal{C})$, 2of6 applied to

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[d, "\overline{f}"] & \\ & \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

means $Iso(\mathcal{C}) \subseteq W$.

Def: $(M, \mathrm{Iso}(M))$ is called the minimal homotopical category of M.

Eg $(\mathrm{Top}, \{\text{weak equiv}\})$ is homotopical.

Pf: Follows from the functorality of π_n, and the previous example. $\square$

Eg $(\mathrm{Top}, \{\text{homotopy equiv}\})$ is homotopical.

Pf: Same proof as with $(M, \mathrm{Iso}(M))$ $\square$

Def A functor $F: (A, W_A) \to (B, W_B)$ is homotopical if $F(W_A) \subseteq F(W_B)$.

Eg π_n, H_n.

Non-Example

Let $F: A \to B$ be additive, A, B abelian, $F_*: Ch_{\geq 0}(A) \to Ch_{\geq 0}(B)$.

Let $W = \{\text{quasi-isomorphism}\}$
$= \{f: C \to D : f_*: H_n(C) \xrightarrow{\cong} H_n(D)\}$.

Let $f: C \to D$ be.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}/2 \arrow[r, hook] \arrow[d] & \mathbb{Z}/4 \arrow[r, "0"] \arrow[d, two heads] & \mathbb{Z}/2 \arrow[r, hook] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d, two heads] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Let $F: Ab \to Ab$ be $Hom_{\mathbb{Z}}(\mathbb{Z}/2, -)$.

__Claim:__ $F(f)$ is not a quasi-iso.

__Pf:__ Board exercise!

(Applying $Hom_{\mathbb{Z}}(\mathbb{Z}/2, -)$ to

$\mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ gives

$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$, which after

homology does not give an iso.)

__Non-Eg__

Let $\underline{Top}^D$ be a homotopical category.

Let $\operatorname{colim}: Top^D \to Top$.

__Claim:__ colim is not homotopical.

__Pf:__ Board exercise!

((Consider

$$\begin{tikzcd}
D^{n+1} \arrow[d, "\simeq"] & S^n \arrow[l, hook'] \arrow[r, hook] \arrow[d, "1"] & D^{n+1} \arrow[d, "\simeq"] \\
* & S^n \arrow[l] \arrow[r] & *
\end{tikzcd}$$

$\operatorname{colim}(D^{n+1} \hookleftarrow S^n \hookrightarrow D^{n+1}) = S^{n+1}$, but

$\operatorname{colim}(* \leftarrow S^n \rightarrow *) = *$, & $S^n \not\simeq *$.)

- We want to find the closest homotopical approximation to an arbitrary $F:(A,W_A)\to(B,W_B)$. How do we make one?
- Idea: Kan extensions.

<u>Def</u> Let $F:(A,M_A)\to(B,M_B)$ be a functor, & $\mathrm{Ho}M$, $\mathrm{Ho}N$ be the localizations of M, N.

A total left derived functor LF of a functor F is a right Kan extension $\mathrm{Ran}_\gamma \delta F$

$$
\begin{tikzcd}
M \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr] & N \arrow[d, "\delta"] \\
\mathrm{Ho}M \arrow[r, dashed, "LF"'] \arrow[ur, phantom, "\Uparrow"] & \mathrm{Ho}N
\end{tikzcd}
$$

<u>Def:</u> A left derived functor of $F:M\Rightarrow N$ is a homotopical functor $\mathbb{L}F: M\to N$ and a n.t

$\lambda : \mathbb{L}F \to F$ s.t.

$\delta\lambda : \delta \cdot \mathbb{L}F \Rightarrow \delta F$ is a total left derived functor of F. (after lifting).

<u>References:</u> Riehl, "Categorical homotopy theory" (2014)