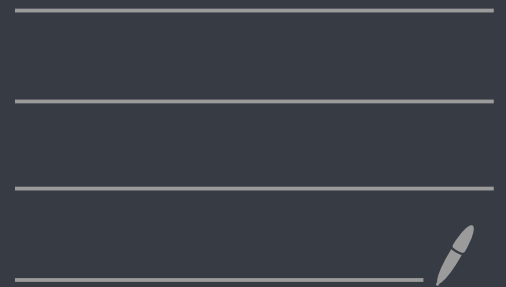


Homotopical Categories II

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Homotopical Categories II

Last time:

Homotopical category (M, W) where

$W \subseteq M$ is a subcat. st:

— $\text{obj } W = \text{obj } M$

— W satisfies the 2 of 6 condition.

($gf, hg \in W \Rightarrow f, g, h, hg \circ f \in W$)

Ex $(C, \text{hom}(C))$ (minimal homotopical)

Ex $(\text{Top}, \{ \text{htpy equiv} \})$

Homotopical functor := preserves weak equiv.

Ex Π_n, H_n

Non-Ex colim: $\text{Top}^D \rightarrow \text{Top}$

Non-Ex Induced $F_{\#}: \text{Ch}(Ab) \rightarrow \text{Ch}(Ab)$

where $W = \{ \text{quasi-isomorphisms} \}$

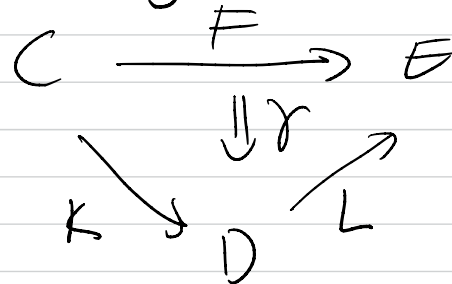
Idea: Use Kan extensions to approximate non-homotopical functors.

Def: Given $F: C \rightarrow E, K: C \rightarrow D,$

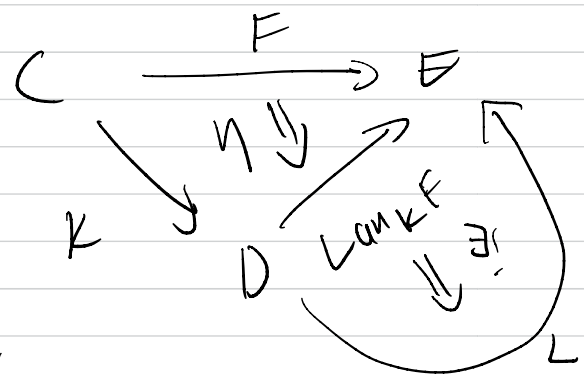
a left Kan extension is a pair

$(Lan_K F: D \rightarrow E, \eta: F \rightarrow Lan_K F \circ K)$

s.t. any (L, γ) satisfying



factors uniquely



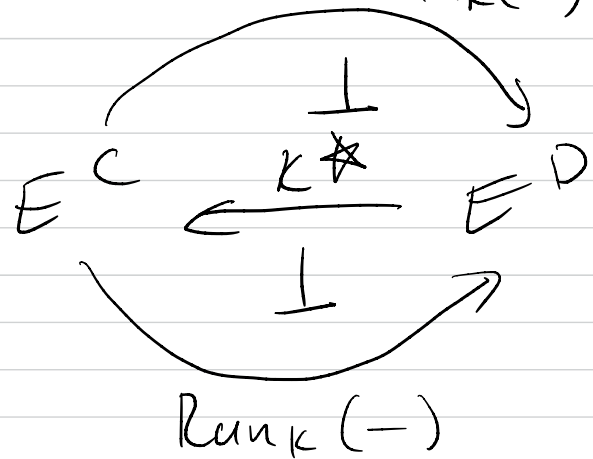
Right Kan extension:
natural trans. go in other direction

Prop: Let $k: C \rightarrow D$ be fixed.

If $\text{Lan}_k F$ & $\text{Ran}_k F$ exist for every $F \in E^C$ then:

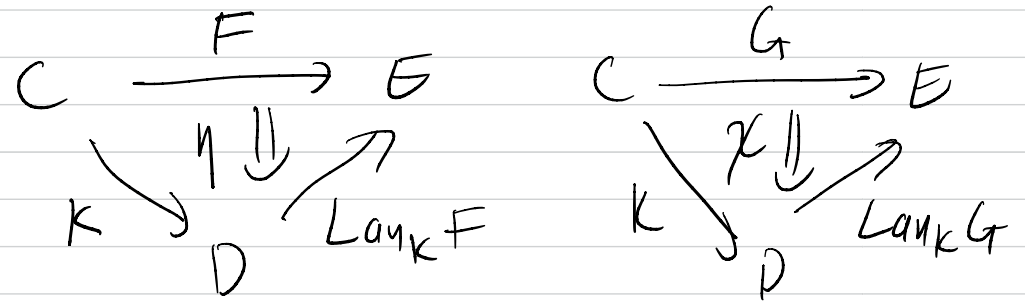
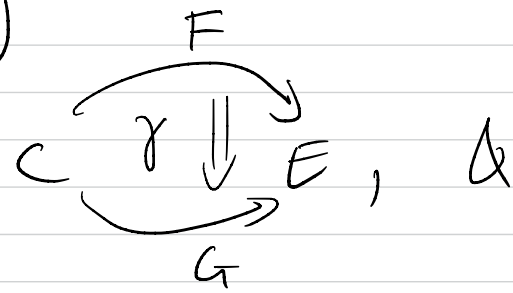
(1) $\text{Lan}_k(-): E^C \rightarrow E^D$ is a functor.

(2) We have $\text{Lan}_k(-)$

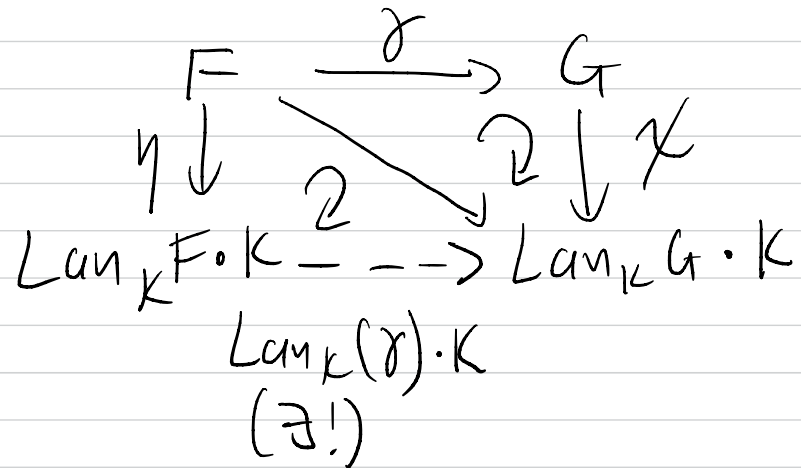


Pf: (1)

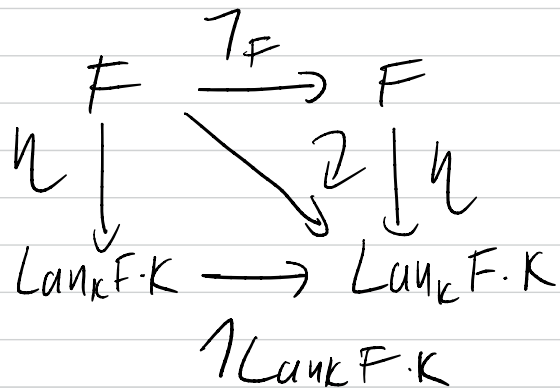
Let



Define $\text{Lan}_k(\gamma)$ by:

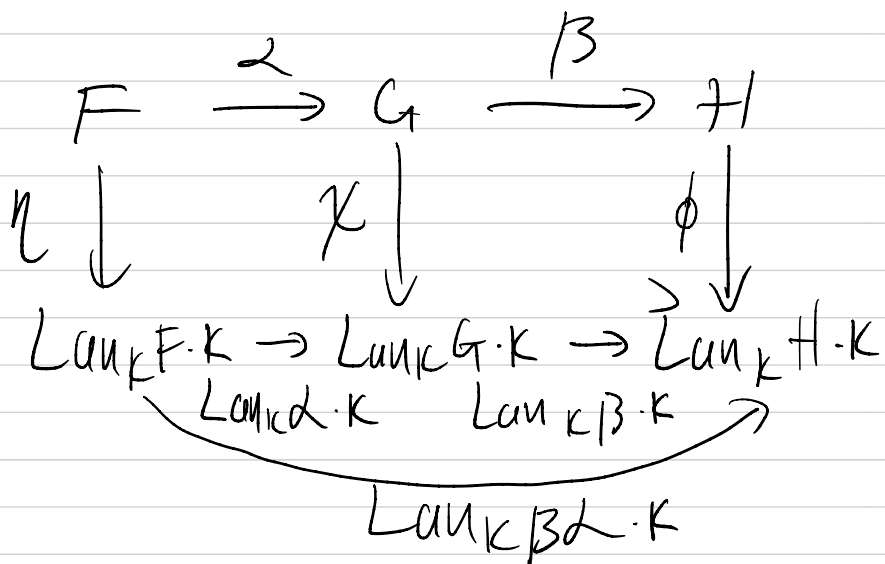


To check $\text{Lan}_k(-)$ is a functor:



is a commutative diagram.

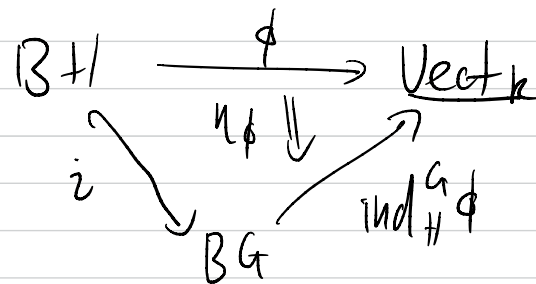
So uniqueness $\Rightarrow \text{Lan}_k(1_F) = 1_{\text{Lan}_k F}$.



A diagram chase shows $\text{Lan}_k \beta = \text{Lan}_k \beta \cdot \text{Lan}_k \alpha$.

(2): See ^{Riehl,} Category Theory in Context, p.142. \square

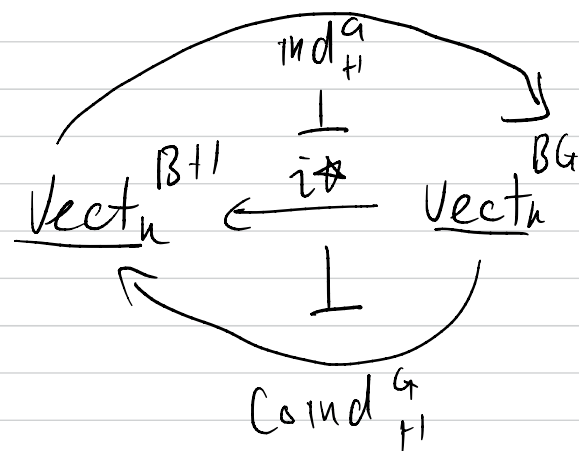
Eg Let $H \leq G$ be groups.



$\text{ind}_H^G \phi$ is a $\text{Lan}_i \phi$

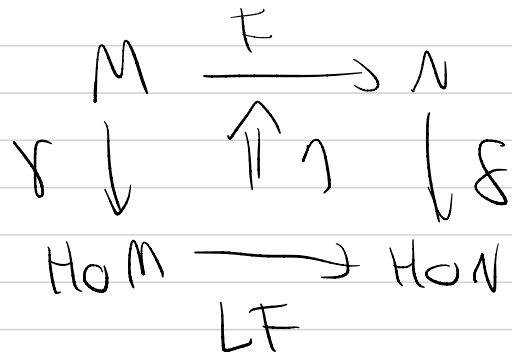
Similarly $\text{coind}_H^G \phi$ is a $\text{Ran}_i \phi$.

Hence



Def: Let M, N be homotopical.

A total left derived functor $\tilde{LF}: HoM \rightarrow HoN$ is a right Kan extension $Ran_{\gamma} \delta F$

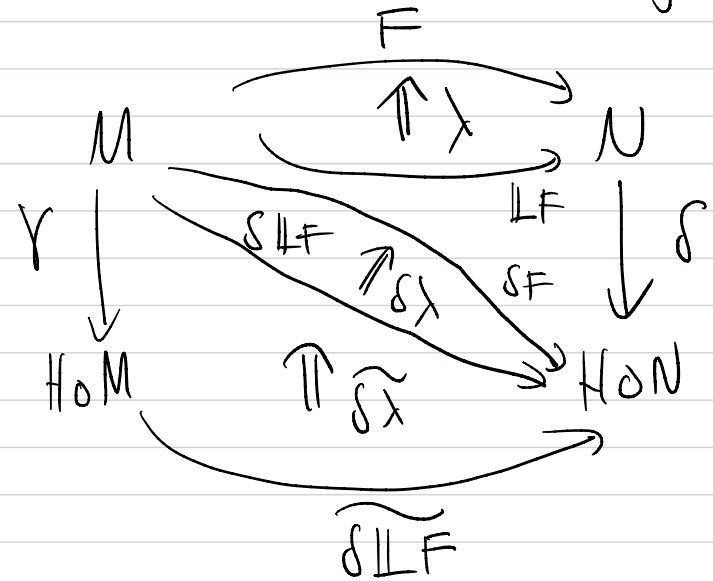


Rm: \tilde{LF} corresponds to a homotopical functor

$$LF: M \rightarrow HoN.$$

Def: A left derived functor of $F: M \rightarrow N$ is a homotopical functor $LF: M \rightarrow N$ with a n.t. $\chi: LF \rightarrow F$ s.t.

How is this defined? $\delta\chi: \delta LF \rightarrow \delta F$ is a total left derived functor of F .



Def: A left deformation on a homotopical category M is an endofunctor Q with a natural weak equivalence

$$q: Q \xrightarrow{\sim} 1$$

Lemma: Left deformations are homotopical.

Pf: For a homotopical (M, W) let $x \xrightarrow{f} y \in W$.

Then by definition of q ,

$$\begin{array}{ccc} Qx & \xrightarrow{Qf} & Qy \\ q_x \downarrow & \cong & \downarrow q_y \\ x & \xrightarrow{f} & y \end{array} \quad \begin{array}{l} f q_x \in W \Rightarrow q_y Qf \in W \\ 2 \text{ of } 3 \Rightarrow Qf \in W. \quad \square \end{array}$$

Def: A left deformation of $F: M \rightarrow N$ is a left deformation Q of M s.t. M is homotopical on a full subcategory containing the image of Q .

Thm (Dwyer, Hirschhorn, Kan, Smith '04)

If $F: M \rightarrow N$ has a left deformation Q , then FQ is a left derived functor of F .

Pf: We want to show (LFQ, LFq) satisfy the universal property of total left derived functors.

Let $G: M \rightarrow \text{HoN}$ be homotopical, and

$$\gamma: G \rightarrow \text{SF}$$

By definition of γ being natural,

$$\begin{array}{ccc} GQx & \xrightarrow{Gq_x} & Gx \\ \gamma_{Qx} \downarrow & \wr & \downarrow \gamma_x \\ \text{SF}Qx & \xrightarrow{\text{SF}q_x} & \text{SF}x \end{array}$$

Since the q_x are weak eqs, Gq_x are ISOs.

$$\text{Hence } \gamma_x = \text{SF}q_x \circ \gamma_{Qx} \circ (Gq_x)^{-1},$$

meaning γ factors through $\text{SF}q$.

To show the factoring is unique, suppose $\gamma = \text{SF}q \circ \gamma'$.

$$\text{Then } \gamma_Q = \text{SF}q_Q \circ \gamma'_Q.$$

Since F is a left deformation, F sends the weak equivalences of q_Q to weak equivalences, hence $\text{SF}q_Q$ is a natural ISO. Hence $\gamma'_Q = (\text{SF}q_Q)^{-1} \circ \gamma_Q$ is determined by γ_Q . Uniqueness in general follows from

$$\begin{array}{ccc} GQ & \xrightarrow{\gamma'_Q} & \text{SF}Q^2 \\ Gq \downarrow & \wr & \downarrow \text{SF}Qq \\ G & \xrightarrow{\gamma'} & \text{SF}Q \end{array}$$

since both Gq and $\text{SF}Qq$ are natural isomorphisms. \square

Coro: Given an additive

$$F: \text{Mod}_R \rightarrow \text{Mod}_S,$$

$$F_*: \text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(S) \text{ with}$$

weak equivalences quasi-isos
has a left derived functor.

Rm: The classical left
derived functor is the
composition

$$\text{Mod}_R \xrightarrow{\text{deg}_0} \text{Ch}_{\geq 0}(R) \xrightarrow{\mathbb{L}F_*} \text{Ch}_{\geq 0}(S) \xrightarrow{H_0} \text{Mod}_S.$$

Pf: Quasi-isos on non-negatively
graded projective chain complexes
are homotopy equivalences.

i.e. F_* is homotopical on the
image of the functor Q
which takes projective
resolutions.

Hence by the Thm, $\mathbb{L}F_* = F_*Q$
is a left derived functor.