Homotopical Categories II

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Last time:

Homotopical category (M, W) where $W \subseteq M$ is a subcat. st:

— $\mathrm{obj}\, W = \mathrm{obj}\, M$

— W satisfies the 2 of 6 condition.

$(gf, hg \in W \Rightarrow f, g, h, hgf \in W)$

Eg $(C, \mathrm{Iso}(C))$ (minimal homotopical)

Eg $(\mathrm{Top}, \{\text{hpty equivs}\})$

Homotopical functor := preserves weak equiv.

Eg π_n, H_n

Non-Eg colim: $\mathrm{Top}^D \to \mathrm{Top}$

Non-Eg Induced $F_*: Ch(Ab) \to Ch(Ab)$

where $W = \{\text{quasi-isomorphisms}\}$.

Idea: Use Kan extensions to approximate non-homotopical functors.

Def: Given $F: C \to E$, $K: C \to D$, a left Kan extension is a pair

$(\mathrm{Lan}_K F: D \to E,\ \eta: F \Rightarrow \mathrm{Lan}_K F \circ K)$

s.t any (L, γ) satisfying

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "K"'] & \Downarrow \gamma & E \\ & D \arrow[ur, "L"'] & \end{tikzcd}$$

factors uniquely

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "K"'] & \eta \Downarrow & E \\ & D \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right=70, "L"'] \arrow[ur, phantom, bend right=35, "\Downarrow \exists!"] & \end{tikzcd}$$

Right Kan extension:

natural trans. go in other direction

Prop: Let $K : C \to D$ be fixed.

If $\mathrm{Lan}_K F$ & $\mathrm{Ran}_K F$ exist for every $F \in E^C$ then:

(1) $\mathrm{Lan}_K(-) : E^C \to E^D$ is a functor.

(2) We have

$$\begin{tikzcd}
E^C \arrow[r, bend left=60, "\mathrm{Lan}_K(-)"] \arrow[r, bend right=60, "\mathrm{Ran}_K(-)"'] \arrow[r, phantom, bend left=30, "\perp"] \arrow[r, phantom, bend right=30, "\perp"] & E^D \arrow[l, "K^*"']
\end{tikzcd}$$

.

Pf: (1)

Let
$$\begin{tikzcd}
C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\gamma \Downarrow"] & E
\end{tikzcd}$$
, &

$$\begin{tikzcd}
C \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\eta \Downarrow"] & E \\
& D \arrow[ur, "\mathrm{Lan}_K F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "G"] \arrow[dr, "K"'] & \arrow[d, phantom, "\chi \Downarrow"] & E \\
& D \arrow[ur, "\mathrm{Lan}_K G"'] &
\end{tikzcd}$$

Define $\mathrm{Lan}_K(\gamma)$ by:

$$\begin{tikzcd}
F \arrow[r, "\gamma"] \arrow[d, "\eta"'] \arrow[dr, "\circlearrowleft", "\circlearrowleft"'] & G \arrow[d, "\chi"] \\
\mathrm{Lan}_K F \cdot K \arrow[r, dashed, "\mathrm{Lan}_K(\gamma)\cdot K \; (\exists!)"'] & \mathrm{Lan}_K G \cdot K
\end{tikzcd}$$

To check $\mathrm{Lan}_K(-)$ is a functor:

$$\begin{tikzcd}
F \arrow[r, "1_F"] \arrow[d, "\eta"'] \arrow[dr] & F \arrow[d, "\eta"] \\
\mathrm{Lan}_K F \cdot K \arrow[r, "1_{\mathrm{Lan}_K F \cdot K}"'] & \mathrm{Lan}_K F \cdot K
\end{tikzcd}$$

is a commutative diagram.

So uniqueness $\Rightarrow \mathrm{Lan}_K(1_F) = 1_{\mathrm{Lan}_K F}$.

$$\begin{tikzcd}
F \arrow[r, "\alpha"] \arrow[d, "\eta"'] & G \arrow[r, "\beta"] \arrow[d, "\chi"'] & H \arrow[d, "\phi"'] \\
\mathrm{Lan}_K F \cdot K \arrow[r, "\mathrm{Lan}_K \alpha \cdot K"'] \arrow[rr, bend right, "\mathrm{Lan}_K \beta\alpha \cdot K"'] & \mathrm{Lan}_K G \cdot K \arrow[r, "\mathrm{Lan}_K \beta \cdot K"'] & \mathrm{Lan}_K H \cdot K
\end{tikzcd}$$

A diagram chase shows

$\mathrm{Lan}_K \beta\alpha = \mathrm{Lan}_K \beta \cdot \mathrm{Lan}_K \alpha$.

(2): See Riehl, Category Theory in Context, p.192. □

Eg Let $H \leq G$ be groups.

$$\begin{tikzcd}
BH \arrow[rr, "\phi"] \arrow[dr, "i"'] \arrow[rr, phantom, shift right=3ex, "\Downarrow \eta_\phi"] & & \underline{\mathrm{Vect}_k} \\
& BG \arrow[ur, "\mathrm{ind}_H^G \phi"'] &
\end{tikzcd}$$

$\mathrm{ind}_H^G \phi$ is a $\mathrm{Lan}_i \phi$

Similarly $\mathrm{coind}_H^G \phi$ is a $\mathrm{Ran}_i \phi$.

Hence

$$\begin{tikzcd}
\underline{\mathrm{Vect}_k}^{BH} \arrow[r, bend left=60, "\mathrm{ind}_H^G"] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & \underline{\mathrm{Vect}_k}^{BG} \arrow[l, "i^*"'] \arrow[l, bend left=60, "\mathrm{coind}_H^G"]
\end{tikzcd}$$

__Def:__ Let M, N be homotopical. A total left derived functor $\widetilde{LF}: \mathrm{Ho}M \to \mathrm{Ho}N$ is a right Kan extension $\mathrm{Ran}_\gamma \delta F$

$$\begin{tikzcd}
M \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow \eta"] & N \arrow[d, "\delta"] \\
\mathrm{Ho}M \arrow[r, "LF"'] & \mathrm{Ho}N
\end{tikzcd}$$

__Rm:__ $\widetilde{LF}$ corresponds to a homotopical functor $LF: M \to \mathrm{Ho}N$.

__Def:__ A left derived functor of $F: M \to N$ is a homotopical functor $\mathbb{L}F: M \to N$ with a n.t. $\lambda: \mathbb{L}F \to F$ s.t. $\boxed{\widetilde{\delta\lambda} : \widetilde{\delta\mathbb{L}F} \to \delta F}$ is a total left derived functor of F.

How is this defined?

$$\begin{tikzcd}
M \arrow[r, bend left=40, "F"] \arrow[r, bend right=20, "\mathbb{L}F"'] \arrow[r, phantom, "\Uparrow \lambda"] \arrow[d, "\gamma"'] \arrow[dr, bend left=10, "\delta F"] \arrow[dr, bend right=10, "\delta\mathbb{L}F"'] \arrow[dr, phantom, "\Uparrow \delta\lambda"] & N \arrow[d, "\delta"] \\
\mathrm{Ho}M \arrow[r, bend right=40, "\widetilde{\delta\mathbb{L}F}"'] \arrow[r, phantom, "\Uparrow \widetilde{\delta\lambda}"] & \mathrm{Ho}N
\end{tikzcd}$$

Def: A left deformation on a homotopical category M is an endofunctor Q with a natural weak equivalence

$$q : Q \xrightarrow{\sim} 1$$

Lemma: Left deformations are homotopical.

Pf: For a homotopical (M, W) let $x \xrightarrow{f} y \in W$.

Then by definition of q,

$$\begin{tikzcd} Qx \arrow[r, "Qf"] \arrow[d, "q_x"'] \arrow[dr, phantom, "\circlearrowright"] & Qy \arrow[d, "q_y"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

$f q_x \in W \Rightarrow q_y Qf \in W$

2 of 3 $\Rightarrow Qf \in W$. □

Def: A left deformation of $F : M \to N$ is a left deformation Q of M s.t. M is homotopical on a full subcategory containing the image of Q.

Thm (Dwyer, Hirschhorn, Kan, Smith '04)

If $F : M \to N$ has a left deformation Q, then FQ is a left derived functor of F.

Pf: We want to show $(\delta FQ, \delta Fq)$ satisfy the universal property of total left derived functors.

Let $G: M \to HoN$ be homotopical, and $\gamma: G \to \delta F$.

By definition of γ being natural,

$$\begin{tikzcd}
GQx \arrow[r, "Gq_x"] \arrow[d, "\gamma_{Qx}"'] \arrow[dr, phantom, "\circlearrowright \forall x"] & Gx \arrow[d, "\gamma_x"] \\
\delta FQx \arrow[r, "\delta Fq_x"'] & \delta Fx
\end{tikzcd}$$

Since the q_x are weak equivs, Gq_x are isos.

Hence $\gamma_x = \delta F q_x \circ \gamma_{Qx} \circ (Gq_x)^{-1}$,

meaning γ factors through δFq.

To show the factoring is unique, suppose $\gamma = \delta Fq \circ \gamma'$.

Then $\gamma_Q = \delta F q_Q \circ \gamma'_Q$.

Since F is a left deformation, F sends the weak equivalences of q_Q to weak equivalences, hence $\delta F q_Q$ is a natural iso. Hence $\gamma'_Q = (\delta F q_Q)^{-1} \circ \gamma_Q$ is determined by γ_Q. Uniqueness in general follows from

$$\begin{tikzcd}
GQ \arrow[r, "\gamma'_Q"] \arrow[d, "Gq"'] \arrow[dr, phantom, "\circlearrowright"] & \delta FQ^2 \arrow[d, "\delta FQq"] \\
G \arrow[r, "\gamma'"'] & \delta FQ
\end{tikzcd}$$

since both Gq and δFQq natural isomorphisms. □

Coro: Given an additive $F: \mathrm{Mod}_R \to \mathrm{Mod}_S$, $F_*: Ch_{\geq 0}(R) \to Ch_{\geq 0}(S)$ with weak equivalences quasi-isos has a left derived functor.

Pf: Quasi-isos on non-negatively graded *projective* chain complexes are homotopy equivalences. i.e. F_* is homotopical on the image of the functor Q which takes projective resolutions. Hence by the Thm, $\mathbb{L}F_* = F_*Q$ is a left derived functor.

Rm: The classical left derived functor is the composition

$$\mathrm{Mod}_R \xrightarrow{\deg_0} Ch_{\geq 0}(R) \xrightarrow{\mathbb{L}F_*} Ch_{\geq 0}(S) \xrightarrow{H_0} \mathrm{Mod}_S.$$