Proving functions are equal or different

In mathematics, two functions are considered equal if they have the same domain and take the same value at each point in the domain.

Example: Show that $f(x) = 2x$ and $g(x) = 3x$ are different functions.

Proof: Since $f(1) = 2 \neq 3 = g(1)$, f and g take different values at 1. Therefore $f \neq g$.

Sometimes it is not as obvious when two functions are equal:

Example: Let

$$f(x) = \frac{x^3 - x}{x + 1}, \qquad g(x) = x^2 - x$$

have domain positive real numbers. Is $f = g$?

Solution: Note

$$f(x) = \frac{x^3 - x}{x + 1} = \frac{x(x^2 - 1)}{x + 1} = \frac{x(x + 1)(x - 1)}{x + 1} = x(x - 1) = x^2 - x = g(x).$$

Problems

Are the following functions equal? Justify your answers. In particular, if the functions are not equal, prove this by finding a point in the domain where the functions take different values.

1. $f(x) = x^2 - 2x - 8, \qquad g(x) = (x + 4)(x - 2).$

2. $f(x) = x^2 - 2x - 3, \qquad g(x) = (x + 1)(x - 3).$

3. $f(x) = x^2 + 1, \qquad g(x) = (x + 1)^2.$

4. $f(x) = x^3 - 8, \qquad g(x) = (x-2)^3.$

5. $f(x) = \sqrt[3]{x^3 + 3x^2 + 3x + 1}, \qquad g(x) = x + 1.$

6. $f(x) = \sqrt[3]{x^3 + 1}, \qquad g(x) = \sqrt[3]{x^3} + \sqrt[3]{1}.$

7. $f(x) = \sin 2x, \qquad g(x) = 2 \sin x.$

8.
$$f(x) = x^3 - x^2 - x + 1, \qquad g(x) = \begin{cases} (x^4 - 2x^2 + 1)/(x+1) & x \neq 0 \\ 0 & x = -1. \end{cases}$$

9.
$$f(x) = x^2 + 2, \qquad g(x) = \begin{cases} (x^3 - 1)/(x-1) & x \neq 1 \\ 3 & x = 1 \end{cases}.$$

$$f(x) = x^4 + x^3 + x^2 + x + 1, \qquad g(x) = \begin{cases} (x^5 - 1)/(x-1) & x \neq 1 \\ 5 & x = 1 \end{cases}.$$

10. $f(x, y) = x^2 - y^2, \qquad g(x, y) = (x - y)(x + y).$

11. $f(x, y) = (x - y)^2(x + y)^2, \qquad g(x, y) = x^4 - y^4.$

12. $f(x, y) = 10^x + 10^y, \qquad g(x, y) = 10^{x+y}.$

13. $f(x, y) = \frac{1}{x+y}, \qquad g(x, y) = \frac{1}{x} + \frac{1}{y}.$

14. $f(x, y) = \cos(x + y), \qquad g(x, y) = \cos x + \cos y.$

15. $f(x, y, z) = (x + y)^z, \qquad g(x, y, z) = x^z + y^z.$