Problems

Are the following functions equal? Justify your answers. In particular, if the functions are not equal, prove this by finding a point in the domain where the functions take different values.

1. $f(x) = x^2 - 2x - 8, \qquad g(x) = (x+4)(x-2).$
2. $f(x) = x^2 - 2x - 3, \qquad g(x) = (x+1)(x-3).$
3. $f(x) = x^2 + 1, \qquad g(x) = (x+1)^2.$
4. $f(x) = x^3 - 8, \qquad g(x) = (x-2)^3.$
5. $f(x) = \sqrt[3]{x^3 + 3x^2 + 3x + 1}, \qquad g(x) = x + 1.$
6. $f(x) = \sqrt[3]{x^3 + 1}, \qquad g(x) = \sqrt[3]{x^3} + \sqrt[3]{1}.$
7. $f(x) = \sin 2x, \qquad g(x) = 2\sin x.$
8. $$f(x) = x^3 - x^2 - x + 1, \qquad g(x) = \begin{cases} (x^4 - 2x^2 + 1)/(x+1) & x \neq -1 \\ 0 & x = -1. \end{cases}$$
9. $$f(x) = x^2 + 2, \qquad g(x) = \begin{cases} (x^3 - 1)/(x-1) & x \neq 1 \\ 3 & x = 1 \end{cases}.$$
   $$f(x) = x^4 + x^3 + x^2 + x + 1, \qquad g(x) = \begin{cases} (x^5 - 1)/(x-1) & x \neq 1 \\ 5 & x = 1 \end{cases}.$$
10. $f(x, y) = x^2 - y^2, \qquad g(x, y) = (x-y)(x+y).$
11. $f(x, y) = (x-y)^2(x+y)^2, \qquad g(x, y) = x^4 - y^4.$
12. $f(x, y) = 10^x + 10^y, \qquad g(x, y) = 10^{x+y}.$
13. $f(x, y) = \frac{1}{x+y}, \qquad g(x, y) = \frac{1}{x} + \frac{1}{y}.$
14. $f(x, y) = \cos(x+y), \qquad g(x, y) = \cos x + \cos y.$
15. $f(x, y, z) = (x+y)^z, \qquad g(x, y, z) = x^z + y^z.$

Solutions

1. $f(1) = -9 \neq -5 = g(1)$, so $f \neq g$.
2. Expanding $(x+1)(x-3)$ gives $x^2 - 2x - 3$, so $f = g$.
3. $f(2) = 5 \neq 9 = g(2)$, so $f \neq g$.
4. $f(1) = -7 \neq -1 = g(1)$, so $f \neq g$.
5. $x^3 + 3x^2 + 3x + 1 = (x+1)^3$, so
$$f(x) = \sqrt[3]{x^3 + 3x^2 + 3x + 1} = \sqrt[3]{(x+1)^3} = x + 1 = g(x).$$
6. $f(1) = \sqrt[3]{2} \neq 2 = g(1)$, so $f \neq g$.
7. $f(\pi/2) = 0 \neq 2 = g(\pi/2)$, so $f \neq g$.
8. Note that
$$x^4 - 2x^2 + 1 = (x^2-1)^2 = \left((x+1)(x-1)\right)^2 = (x+1)^2(x-1)^2.$$
So when $x \neq -1$,
$$\frac{x^4 - 2x^2 + 1}{x+1} = (x+1)(x-1)^2 = (x^2-1)(x-1) = x^3 - x^2 - x + 1,$$
meaning $f(x) = g(x)$ for $x \neq -1$. When $x = -1$, $f(-1) = (-1)^3 - (-1)^2 - (-1) + 1 = 0 = g(0)$. Therefore f and g agree at all points in their domain, so $f = g$.
9. (a) $f(2) = 6 \neq 7 = g(2)$, so $f \neq g$.
   (b) Dividing $x^5 - 1$ by $x - 1$ shows that $(x-1)(x^4 + x^3 + x^2 + x + 1) = x^5 - 1$. Hence $f(x) = g(x)$ for $x \neq 1$, and when $x = 1$, $f(1) = 5 = g(1)$. Therefore $f = g$.
10. Expanding $(x-y)(x+y)$ shows that $f = g$.
11. $f(2,1) = 1^2 \cdot 3^2 = 9 \neq 15 = 2^4 - 1^4 = g(2,1)$, so $f \neq g$.
12. $f(0,1) = 10^0 + 10^1 = 1 + 10 = 11 \neq 10 = g(0,1)$, so $f \neq g$.
13. $f(-2,4) = \frac{1}{2} \neq -\frac{1}{4} = g(-2,4)$, so $f \neq g$.
14. $f(0,0) = 1 \neq 2 = g(0,0)$, so $f \neq g$.
15. $f(1,1,2) = 2^2 = 4 \neq 2 = 1^2 + 1^2 = g(1,1,2)$, so $f \neq g$.