Quiz and Solutions

Problems

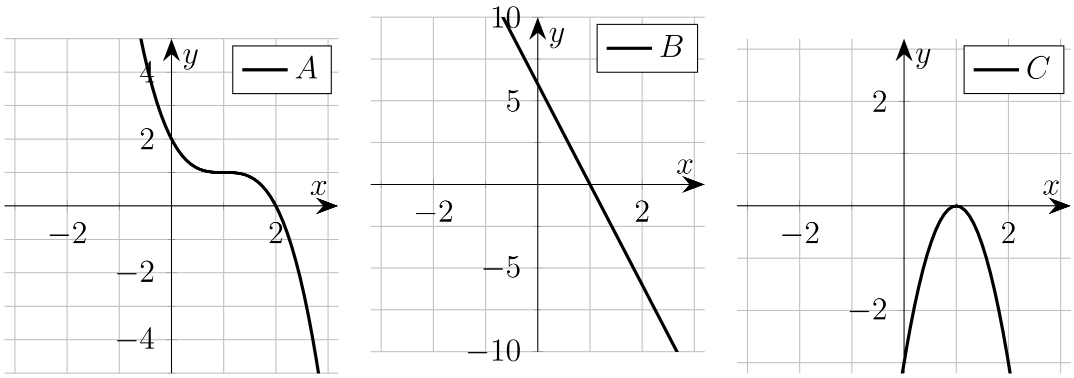
1. Evaluate
$$\lim_{\theta \to 0} \frac{\theta}{\tan \theta}$$
if it exists. If it does not, explain why.

2. The tangent to the curve $y = x^2$ at $x = 10$ intersects the x-axis. Find this intersection point.

3. Let
$$f(x) = \begin{cases} x + 1 & x < 0 \\ 1 - x^2 & x \geq 0. \end{cases}$$
Is f differentiable at $x = 0$? Why / why not?

4. Does $(pq)' = p'q'$ for all polynomials $p(x), q(x)$? If not, give a pair of polynomials $(a(x), b(x))$ for which $(ab)' \neq a'b'$.

5. A polynomial $p(x)$, as well as its derivative $p'(x)$ and second derivative $p''(x)$ are plotted below. Match p, p', p'' with A, B, C.



Solutions

1. Using the identity $\tan \theta = \sin \theta / \cos \theta$, we can do some rearranging:
$$\frac{\theta}{\tan \theta} = \frac{\theta}{\frac{\sin \theta}{\cos \theta}} = \frac{\theta \cos \theta}{\sin \theta} = \cos \theta \cdot \frac{1}{\frac{\sin \theta}{\theta}}.$$
Since $\lim_{\theta \to 0} \sin \theta / \theta = 1 \neq 0$,
$$\lim_{\theta \to 0} \frac{1}{\frac{\sin \theta}{\theta}} = \frac{1}{1} = 1.$$

The function $\cos\theta$ is continuous, so $\lim_{\theta\to 0}\cos\theta = \cos(0) = 1$. We can then apply the product limit law to conclude

$$\lim_{\theta\to 0}\left(\cos\theta\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right) = \left(\lim_{\theta\to 0}\cos\theta\right)\left(\lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}\right) = 1\cdot 1 = 1.$$

2. Since $f'(x) = 2x$, $f'(10) = 20$, so the gradient of the tangent line to the curve $y = x^2$ at $x = 10$ is 20.

   A non-vertical line with gradient m passing through a point (x_0, y_0) has equation $y - y_0 = m(x - x_0)$.

   The tangent line touches the curve at $(10, 100)$ and therefore passes through this point. Therefore the equation of the tangent line is $y - 100 = 20(x - 10)$. Expanding the RHS of this equation and rearranging this gives $y = 20x - 100$.

   Every point on the x-axis has a y-value of 0, so to find the intersection point of the tangent line with the x-axis we substitute $y = 0$ into the equation of the line and solve for x:

$$\begin{aligned} &0 = 20x - 100 \\ &\Rightarrow 20x = 100 \\ &\Rightarrow x = 100/20 = 5. \end{aligned}$$

3. Since $(x+1)' = 1$ while $(1 - x^2)' = -2x$, and $1 \neq -2x$ at $x = 0$, we can expect that the derivative will not exist at $x = 0$. We can formally prove this using the limit definition of the derivative and computing left and right-handed limits:

$$\lim_{h\to 0^-}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-}\frac{(1+h) - 1}{h} = \lim_{h\to 0} 1 = 1,$$

$$\lim_{h\to 0^+}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+}\frac{1 - (0+h)^2 - 1}{h} = \lim_{h\to 0^+}\frac{-h^2}{h} = -\lim_{h\to 0^+} h = 0.$$

   Since the left-hand and right-hand limits are not equal, the overall limit does not exist, so f is not differentiable at $x = 0$.

4. Certainly not. Let $a(x) = b(x) = x$. Then

$$(ab)' = (x^2)' = 2x \neq 1 = 1\cdot 1 = a'b'.$$

5. Since B is a plot of a line, B' is constant. Neither A nor C is a constant function, so $B' \neq A$ and $B' \neq C$, which means $B = p''$.

   Note that C has a horizontal tangent line at $x = 1$, so C' must be 0 at $x = 1$. Looking at the plots, we see B is 0 at $x = 1$ but A is non-zero at $x = 1$, so $C' = B$. Hence $C = p'$. It follows that $(A, B, C) = (p, p'', p)$.