Question 1

One of the most important inequalities in mathematics in the Cauchy–Schwarz inequality. In the Euclidean setting, it states that for all vectors $a, b \in R^n$,

$$|a \cdot b| \leq ||a|| \cdot ||b||$$

where the dot on the left represents the dot product and the dot on the right is multiplication of real numbers.

1. Using elementary algebra, prove the Cauchy–Schwarz inequality for vectors in the Euclidean plane (R^2).

2. Using the Cauchy–Schwarz inequality, prove that when $x_1 + x_2 + \ldots + x_n \geq 0$,

$$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \ldots x_n}{n}.$$

3. Prove that if $a + b + c = 1$, then $a^2 + b^2 + c^2 \geq \frac{1}{3}$.

4. Using the Cauchy–Schwarz inequality or otherwise, prove that for $a, b \geq 0$,

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

5. Prove that for any $a \geq 0$,

$$\frac{1+a}{2\sqrt{a}} > 1.$$

   Note: This inequality is a step that appears in some proofs of the Riemann Mapping Theorem[1], one of the most important results in Complex Analysis.

Question 2

For what values of a, b, are the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix},$$

linearly independent?

[1]For example in p.158 of *Functions of One Complex Variable* by John B. Conway